Exercise 67

If $f(x) = (x-a)(x-b)(x-c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

Solution

Recognize that $f'(x)/f(x)$ is the derivative of a logarithm by the chain rule.

$$\begin{aligned}
\frac{f'(x)}{f(x)} &= \frac{d}{dx}\ln f(x) \\
&= \frac{d}{dx}\ln[(x-a)(x-b)(x-c)] \\
&= \frac{d}{dx}[\ln(x-a) + \ln(x-b) + \ln(x-c)] \\
&= \left[\frac{d}{dx}\ln(x-a)\right] + \left[\frac{d}{dx}\ln(x-b)\right] + \left[\frac{d}{dx}\ln(x-c)\right] \\
&= \left[\left(\frac{1}{x-a}\right)\cdot\frac{d}{dx}(x-a)\right] + \left[\left(\frac{1}{x-b}\right)\cdot\frac{d}{dx}(x-b)\right] + \left[\left(\frac{1}{x-c}\right)\cdot\frac{d}{dx}(x-c)\right] \\
&= \left[\left(\frac{1}{x-a}\right)\cdot(1)\right] + \left[\left(\frac{1}{x-b}\right)\cdot(1)\right] + \left[\left(\frac{1}{x-c}\right)\cdot(1)\right] \\
&= \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}
\end{aligned}$$